5. Bruggeman, D. A. G., *Ann. Physik.*, **24**, 636 (1935).
6. Chambre, P. L., and Andreas Acrivos, *J. Appl. Phys.*, **27**, 1322 (1956).
7. Gardner, G. O., and J. Kestin, *Intern. J. Heat Mass Transfer*, **6**, 295 (1963).
8. Gilliland, E. R., *Ind. Eng. Chem.*, **30**, 512 (1938).
9. Hanna, O. T., *Chem. Eng. Sci.*, **17**, 1045 (1962).
10. Hansen, D., paper presented at 67 Ann. Meeting Am. Soc. Testing Material, Chicago (1964).
11. Khokhani, K. H., M.S. thesis, Univ. Missouri, Columbia (1966).
12. Kunii, D., and J. M. Smith, *A.I.Ch.E. J.*, **6**, 71 (1960).
13. Levich, V. G., "Psysicochemical Hydrodynamics," p. 100, Prentice-Hall, Englewood-Cliffs, N. J. (1962).
14. Lighthill, E. N., *Proc. Roy. Soc. London*, **A202**, 359 (1950).
15. Luikov, A. V., *Intern. Chem. Eng.*, **3**, 195-202 (1963).
16. ———, *Intern. J. Heat Mass Transfer*, **6**, 559-570 (1963).
17. ———, "Heat and Mass Transfer in the Drying Process" (in Russian), Gosenergoizdat, Moscow (1956).
18. Morgan, R. P., Ph.D. thesis, Rensselaer Polytech. Inst., Troy, N. Y. (1965).
19. Nissan, A. H., H. H. George, Jr., and T. V. Bolles, *A.I.Ch.E. J.*, **6**, 406-410 (1960).
20. Pearse, J. F., T. R. Oliver, and D. M. Newitt, *Trans. Inst. Chem. Engrs. (London)*, **27**, 1, 9 (1949).
21. Rosner, D. E., *Chem. Eng. Sci.*, **19**, 10 (1963).
22. Shubin, A. S., *Intern. J. Appl. Rad. Isotopes*, **2**, 110 (1957).
23. Smolsky, B. M., and G. T. Sergeev, *Intern. J. Heat Mass Transfer*, **5**, 1015 (1962).
24. Tribus, Max, and J. Klein, "Forced Convection from non-Isothermal Surfaces," Eng. Res. Inst., Univ. Michigan, Ann Arbor (Aug., 1952).

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Heat Transfer and Binary Diffusion with Thermodynamic Coupling in Variable-Property Forced Convection on a Flat Plate

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The variable-property, laminar, boundary-layer equations, which describe simultaneous momentum, heat, and binary mass transfer with thermodynamic coupling,* are analyzed for air flows over a flat plate with the injection of foreign gases through the solid surface. A simplified general treatment of thermodynamic coupling is developed and applied to yield approximate but accurate expressions for evaluating heat transfer rates and adiabatic wall temperatures for injection of hydrogen, helium, and carbon dioxide. This method explains why the driving force based on $(T_w - T_{aw})$ can be used to correlate heat transfer results for situations where diffusion thermo is important. Furthermore, and perhaps most significant, the method provides a simple error estimate for the correlation obtained by using the adiabatic wall temperature.

It is shown that the injection of lightweight gases can significantly reduce viscous dissipation in flows over slender bodies and that diffusion thermo and dissipation effects can be of the same order of magnitude even for reasonably high-velocity flows. These effects are discussed in terms of convenient quantities, called σ functions.

Recent applications, particularly in the aerospace field, have intensified interest in finding suitable means for decreasing the rate of heat transfer from hot fluid layers to solid surfaces. This is primarily because thermally resistant materials for turbine blades, exhaust nozzles, combustion chamber walls, high-speed aircraft, reentry vehicles, etc, may be developed only within certain limits; beyond these a means must be found for maintaining tolerable surface temperatures by reducing heat transfer. Most of the methods proposed for this purpose involve the transfer of mass from or through the solid surface or mass transfer from a liquid film covering the solid surface. Thus a considerable body of literature on the problem of simultaneous heat and mass transfer has arisen and many of the most recent papers have been discussed by Sparrow (*11*) and by Zeh (*21*). These studies are all concerned primarily with heat

* Thermodynamic coupling refers to the Soret (or thermal diffusion) effect, which is the flow of mass caused by a temperature gradient, and the Dufour (or diffusion thermo) effect, which is the flow of heat caused by a concentration gradient.

transfer and have not examined mass transfer effects in nearly as much detail.

It was suggested by a number of workers (for example, 6, 10) that lightweight gases had certain advantages as coolants [although Hurley (10) ranked water vapor above helium but below hydrogen]. Experiments were carried out to verify these predictions by Gollnick (7), Tewfik et al. (18 to 20) and others (references 6 to 11 of reference 1). It was found that theory and experiment agreed well for air injection but not for helium injection.

Indeed, it was found that under certain conditions injection of helium into air had the effect of increasing rather than decreasing the surface temperature. For example, for the case of no net heat transfer to the surface, the surface temperature exceeded the ambient temperature by up to 40°F. (19) or even higher (20) under conditions which precluded aerodynamic heating as the cause (the magnitude of aerodynamic heating effects is discussed later). As remarked by Baron (1), this pointed to incomplete rather than incorrect analysis and in particular incomplete regarding the omission of a mixture phenomenon. Baron was apparently the first to present an analytical study of thermodynamic coupling which reconciled the anomalies between theory and experiment, and found diffusion thermo to be the important omitted effect. There followed a number of analytical papers on the subject by Tewfik (17), Sparrow et al. (11 to 15), and Baron (2). Qualitatively, some of the more important conclusions reached were:

1. Thermal diffusion effects are more important at low mass transfer rates and at high heat transfer rates. Since in the systems studied the concentration of the injected gas went to zero as the interfacial velocity went to zero, thermal diffusion effects were generally found to be negligible.

2. Diffusion thermo effects increase, relatively speaking, as $T_w \rightarrow T_e$, and as the molecular weight of the injected gas decreases relative to that of the ambient gas. Actually, in all of the systems studied in the preceding references, only helium-air and hydrogen-air mixtures exhibited significant Soret-Dufour effects.

3. Diffusion thermo effects on the heat transfer coefficient are generally small if the coefficient is defined in terms of an adiabatic wall temperature.

In the present study, these conclusions are utilized to enable us to treat the effects of diffusion thermo and viscous dissipation in a generalized manner, by developing an approximate but accurate method for determining the heat flux in terms of functions that can be correlated in a relatively simple way. It is shown that the σ function, defined by Equation (39), is sufficiently insensitive to variations in (T_w/T_e) to allow the influence of diffusion thermo to be correlated in a simple and accurate fashion. The insensitivity of the σ function to (T_w/T_e) also makes it possible to develop an error estimate for the calculation of heat transfer rates by using the driving force $(T_w - T_{aw})$ in conjunction with solutions obtained by neglecting diffusion thermo.

EVALUATION OF PHYSICAL PROPERTIES

The study of binary systems is complicated enormously by the fact that fluid properties can vary radically with both temperature and concentration, which in turn markedly affects heat, mass, and momentum transfer. Thus, one must evaluate the transport properties of nonisothermal mixtures; in the present study the method which employs the kinetic theory of gases with the Lennard-Jones model for intermolecular forces was used as outlined by Hirschfelder, Curtiss, and Bird (9).

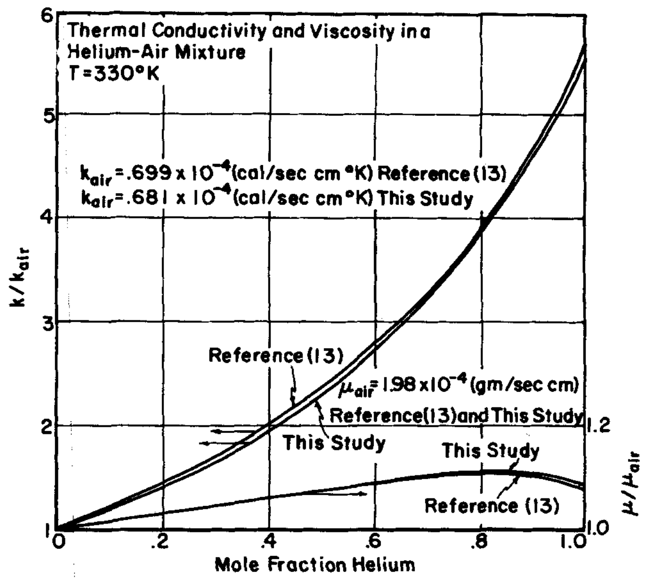

Fig. 1. Comparison of thermal conductivity and viscosity as calculated in present study with those calculated by Eckert et al. (13).

In the course of the investigation it was found that use of the tabulated values of σ_F and ϵ/K_b, evaluated from experimental viscosity data, did not give good results over wide temperature ranges. Furthermore, use of the σ_F evaluated from viscosity data could not be used to predict thermal conductivity data accurately. Consequently it was decided to use the tabulated values of ϵ/K_b to determine separate values of σ_F from experimental viscosity and thermal conductivity data at various temperatures and to fit these data with polynomials in temperature. A program was written to fit orthogonal polynomials of arbitrary order to the data and the coefficients are tabulated elsewhere (21). The temperature dependence of the heat capacity was approximated by orthogonal polynomials and the coefficients are also tabulated in reference 21. The necessary experimental data for all of the calculations was taken from Table A-4 of reference 4.

Sparrow et al. (13) used the Lennard-Jones model in the Chapman-Enskog kinetic theory (3) to evaluate physical properties but in a slightly different manner than is used here. A comparison of the thermal conductivity and viscosity of a helium-air mixture evaluated by the two methods is shown in Figure 1. The maximum deviation between the thermal conductivity curves is about 5%, which can be partially attributed to differences in the pure helium data, since the curves do not meet at $X_{He} = 1$. The viscosity curves are in good agreement except near $X_{He} = 1$ where the deviation is on the order of 5%.

ANALYSIS

It is assumed that pure component A is transmitted from or through a solid surface into an ambient stream of initially pure component B, the latter being air throughout this study. The wall is assumed to be impermeable to component B. For the simultaneous heat and mass transfer systems to be considered the variable-property, boundary-layer equations are

Continuity:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad (1)$$

Momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y} \tag{2}$$

Diffusion:

$$\rho u \frac{\partial \omega_A}{\partial x} + \rho v \frac{\partial \omega_A}{\partial y} = - \frac{\partial j_A}{\partial y} \tag{3}$$

Energy:

$$\rho u C_p \frac{\partial T}{\partial x} + \rho v C_p \frac{\partial T}{\partial y} = - \frac{\partial q}{\partial y} - (C_{p_A} - C_{p_B})\, j_A \frac{\partial T}{\partial y} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \tag{4}$$

with

$$j_A = - \rho D \left(\frac{\partial \omega_A}{\partial y} + \frac{\alpha \omega_A (1 - \omega_A)}{T} \frac{\partial T}{\partial y} \right) \tag{5a}$$

$$q = - k \frac{\partial T}{\partial y} + \alpha R T \frac{M}{M_A M_B} j_A \tag{5b}$$

The boundary conditions for this system of equations are

$$y = 0: u = 0, \omega_A = \omega_{A_w}, T = T_w \tag{6a}$$

$$x \to 0 \text{ and/or } y \to \infty: u = U_e, \omega_A = \omega_{A_e}, T = T_e \tag{6b}$$

Throughout the present study it will be assumed that $\omega_{A_e} = 0$. Also, the solid surface is assumed to be impermeable to component B, so at $y = 0$ we have

$$v_w = - \frac{j_B}{\rho_B} \bigg|_w = \frac{j_A}{\rho - \rho_A} \bigg|_w = - \frac{D}{1 - \omega_A} \left(\frac{\partial \omega_A}{\partial y} + \frac{\alpha \omega_A (1 - \omega_A)}{T} \frac{\partial T}{\partial y} \right) \bigg|_w \tag{7}$$

The continuity equation is identically satisfied by introducing the stream function ψ, defined by

$$\frac{u}{U_e} = \Lambda_\rho^{-1} \frac{\partial \psi}{\partial y_1}, \frac{v}{U_e} = - \Lambda_\rho^{-1} \frac{\partial \psi}{\partial x_1} \tag{8}$$

where Λ represents the ratio of the local value of the subscripted property to its value in the free stream: $\Lambda_\rho = \rho/\rho_e$. Therefore, if one introduces the transformations

$$\eta = \sqrt{\frac{1}{2} \frac{U_e}{\nu_e x}} \int_o^y \Lambda_\mu^{-1} dy \tag{9}$$

$$\psi = \sqrt{2 \frac{U_e x}{\nu_e}} F(\eta) \tag{10}$$

$$\theta = \frac{T - T_e}{T_w - T_e} \tag{11}$$

and

$$\phi = \frac{\omega_A}{\omega_{A_w}} \tag{12}$$

Equations (2), (3), and (4) become

$$\left(\frac{F'}{\Lambda_\rho \Lambda_\mu} \right)'' + F \left(\frac{F'}{\Lambda_\rho \Lambda_\mu} \right)' = 0 \tag{13}$$

$$\left(\frac{\phi'}{N_{Sc}} \right)' + F \phi' + \Delta' = 0 \tag{14}$$

$$\left(\frac{\Lambda_k}{\Lambda_\mu} \theta' \right)' + \Omega \frac{\Lambda_k}{\Lambda_\mu} \theta' + \frac{1}{\frac{T_w}{T_e} - 1} \Gamma' + N_{Pr_e} N_E \left[\left(\frac{F'}{\Lambda_\rho \Lambda_\mu} \right)' \right]^2 = 0 \tag{15}$$

where the primes represent differentiation with respect to η, and

$$\Delta = \frac{\alpha \phi (1 - \omega_{A_w} \phi)}{N_{Sc} \left(1 + \left(\frac{T_w}{T_e} - 1 \right) \theta \right)} \left(\frac{T_w}{T_e} - 1 \right) \theta' \tag{16}$$

$$\Omega = N_{Pr_e} \left\{ \Lambda_C F + \omega_{A_w} \Lambda_{C_{AB}} \left(\frac{\theta'}{N_{Sc}} + \Delta \right) \right\} \frac{\Lambda_\mu}{\Lambda_k} \tag{17}$$

$$\Gamma = \frac{\omega_{A_w} \alpha R \left(1 + \left(\frac{T_w}{T_e} - 1 \right) \theta \right)}{M_A} \frac{M}{M_B} \frac{\mu_e}{k_e} \left(\frac{\phi'}{N_{Sc}} + \Delta \right) \tag{18}$$

$$N_E = \frac{U_e^2}{C_{p_e} (T_w - T_e)} \tag{19}$$

The boundary conditions become

$$\eta = 0: F = F(0), F'(0) = 0, \phi = \theta = 1 \tag{20}$$

$$\eta \to \infty: F' \to \text{unity}, \phi \to 0, \theta \to 0 \tag{21}$$

and the wall concentration is determined from the coupled condition

$$F(0) = \frac{\omega_{A_w}}{1 - \omega_{A_w}} \left[\frac{\phi'(0)}{N_{Sc}} + \Delta_w \right] \tag{22}$$

First, it should be noted that

$$k \sim T^{1/2}, \mu \sim T^{1/2}, D \sim T^{3/2}, \rho \sim T^{-1}$$

Therefore it follows that

$$N_{Sc} \sim T^o, \frac{\Lambda_k}{\Lambda_\mu} \sim T^o, \Lambda_\mu \Lambda_\rho \sim \left(\frac{T}{T_e} \right)^{-1/2}$$

Furthermore, the heat capacity and the thermal diffusion factor are slowly varying functions of temperature. Thus it is seen that the transformations used have minimized temperature dependence of the groupings of physical properties in the resulting differential equations. To a good approximation, one can eliminate the Eckert number as a parameter by defining

$$\theta = \theta_1 + N_E \theta_{VD} \tag{23}$$

where θ_1 and θ_{VD} are assumed to be given by

$$\left(\frac{\Lambda_k}{\Lambda_\mu} \theta_1' \right)' + \Omega \frac{\Lambda_k}{\Lambda_\mu} \theta_1' + \frac{1}{\frac{T_w}{T_e} - 1} \Gamma' = 0 \tag{24}$$

and

$$\left(\frac{\Lambda_k}{\Lambda_\mu} \theta'_{VD} \right)' + \Omega \frac{\Lambda_k}{\Lambda_\mu} \theta'_{VD} + N_{Pr_e} \left\{ \left(\frac{F'}{\Lambda_\rho \Lambda_\mu} \right)' \right\}^2 = 0 \tag{25}$$

In addition, it is assumed that viscous dissipation effects are not too large and therefore the variation of θ_{VD} does not affect the fluid transport properties significantly. In this way we can obtain a reasonable estimate of the extent to which mass transfer alters the amount of energy generation in the boundary layer due to viscous effects without having to perform the rather lengthy calculations necessary to solve this nonlinear system of differential equations for various values of N_E.

Inspection of Equations (13) to (22) clearly indicates that the variable-property problem, which includes the effects of dissipation and thermodynamic coupling, is complicated and involves a large number of parameters. Furthermore, the expressions for the important transport parameters are also very complicated, since the calculated quantities evaluated at the wall

$$N_{Nu} = \frac{hx}{k_e} = \frac{q_w x}{k_e \,\Delta T} = -\sqrt{\frac{1}{2}\frac{xU_e}{\nu_e}}\,\frac{T_w - T_e}{\Delta T}$$

$$\left(\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta'_1(0) + \frac{T_e}{T_w - T_e}\,\Gamma(0) + N_E \frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta'_{VD}(0) \right) \tag{26}$$

$$N_{Sh} = \frac{h_D x}{\rho_e D_e} = \frac{x}{\rho_e D_e}\frac{j_{Aw}}{\Delta\omega} = -\sqrt{\frac{1}{2}\frac{xU_e}{\nu_e}}$$

$$N_{Sc_e}\left(\frac{\phi'(0)}{N_{Sc_w}} + \Delta_w \right) \tag{27}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e U_e^2} = \sqrt{\frac{2\nu e}{U_e x}}\left(\frac{F'}{\Lambda_\rho \Lambda_\mu} \right)'\Bigg|_w \tag{28}$$

are functions of the numerous parameters of the system. Clearly there is a need for a simple method to report results that allows for accurate interpolation and extrapolation.

The correlation of Gross et al. (8) is remarkably simple and yields qualitatively correct results for the flat plate geometry (12) [although Eckert et al. (5) indicated that the results were not generally valid for stagnation flow]. Hence, an approach will be developed in the present study which yields simple results that appear to be accurate over a wide range of parameters.

The works of Baron, Tewfik, and Sparrow et al. have shown in convincing fashion that diffusion thermo can influence heat transfer significantly and that thermal diffusion is usually negligible in this respect. Thus if thermal diffusion is neglected, Equation (24) becomes

$$\left(\frac{\Lambda_k}{\Lambda_\mu}\,\theta_1' \right)' + \Omega \frac{\Lambda_k}{\Lambda_\mu}\,\theta_1' + \frac{1}{\dfrac{T_w}{T_e} - 1}\,\Gamma' = 0 \tag{29}$$

where

$$\Omega = Pr_e \left\{ \Lambda_C F + \omega_{Aw} \Lambda_{CAB} \frac{\phi'}{N_{Sc}} \right\} \frac{\Lambda_\mu}{\Lambda_k} \tag{30}$$

$$\Gamma = \frac{R\mu_e}{k_e}\,\omega_{Aw}\,\alpha \left[1 + \left(\frac{T_w}{T_e} - 1 \right) \theta \right] \frac{M}{M_A M_B}\,\frac{\phi'}{N_{Sc}} \tag{31}$$

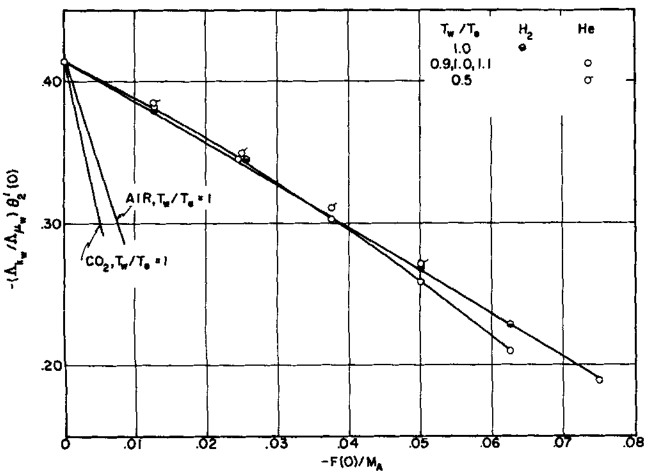

Fig. 2a. $(\Lambda_{kw}/\Lambda_{\mu w})\theta_2'$ **(0) vs.** $F(0)/M_A$ **for injection into air of hydrogen, helium, and carbon dioxide in forced convection flow over a flat plate.** M_A **is molecular weight of pure transpiring gas. Note correlation of hydrogen and helium results.**

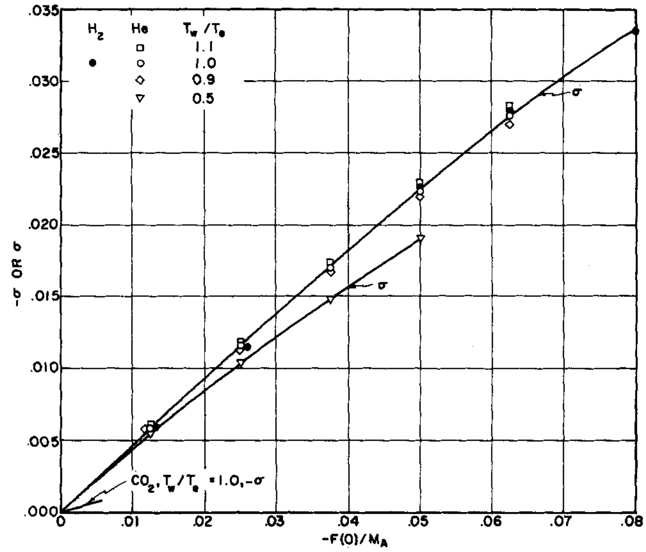

Fig. 2b. σ **vs.** $F(0)/M_A$ **for injection into air of hydrogen, helium, and carbon dioxide in forced convection flow over a flat plate.** M_A **is molecular weight of pure transpiring gas. Note correlation of hydrogen and helium results.**

Obviously, Equation (29) depends on (T_w/T_e) as a parameter but the only strongly temperature-dependent term is in the diffusion thermo term, which is

$$1 + \left(\frac{T_w}{T_e} - 1 \right) \theta_1 = \frac{T}{T_e}$$

(Note that θ_{VD} is not included in this representation of T/T_e.) Consequently, in view of the discussion of property variation with temperature in order to eliminate (T_w/T_e) and (μ_e/k_e) as parameters to a reasonable approximation, one can let

$$\theta_1 = \theta_2 + \frac{1}{2}\left(\frac{T_w}{T_e} + 1 \right) \left(\frac{1}{\dfrac{T_w}{T_e} - 1} \right) \frac{R\mu_e}{k_e}\,\theta_{DT} \tag{32}$$

where θ_2 and θ_{DT} satisfy

$$\left(\frac{\Lambda_k}{\Lambda_\mu}\,\theta_2' \right)' + \Omega \frac{\Lambda_k}{\Lambda_\mu}\,\theta_2' = 0 \tag{33}$$

$$\theta_2(0) = 1,\ \theta_2(\infty) = 0 \tag{34}$$

and

$$\left(\frac{\Lambda_k}{\Lambda_\mu}\,\theta'_{DT} \right)' + \Omega \frac{\Lambda_k}{\Lambda_\mu}\,\theta'_{DT} + \Gamma_2' = 0 \tag{35}$$

$$\theta_{DT}(0) = \theta_{DT}(\infty) = 0 \tag{36}$$

$$\Gamma_2 = \alpha\,\omega_{Aw}\,\frac{M}{M_A M_B}\,\frac{2\left(1 + \left(\dfrac{T_w}{T_e} - 1 \right)\theta_1\right)}{\dfrac{T_w}{T_e} + 1}\,\frac{\phi'}{N_{Sc}} \tag{37}$$

Here the effect of variation in

$$(T/T_e) = 1 + \left(\frac{T_w}{T_e} - 1 \right) \theta_1$$

in the diffusion thermo term has been minimized by dividing it by the average quantity $\left(\dfrac{T_w + T_e}{2T_e} \right)$. For the purpose of evaluating Γ_2 and the physical properties, it is assumed that $\theta_1 \simeq \theta_2$. That is, one solves Equations (13),

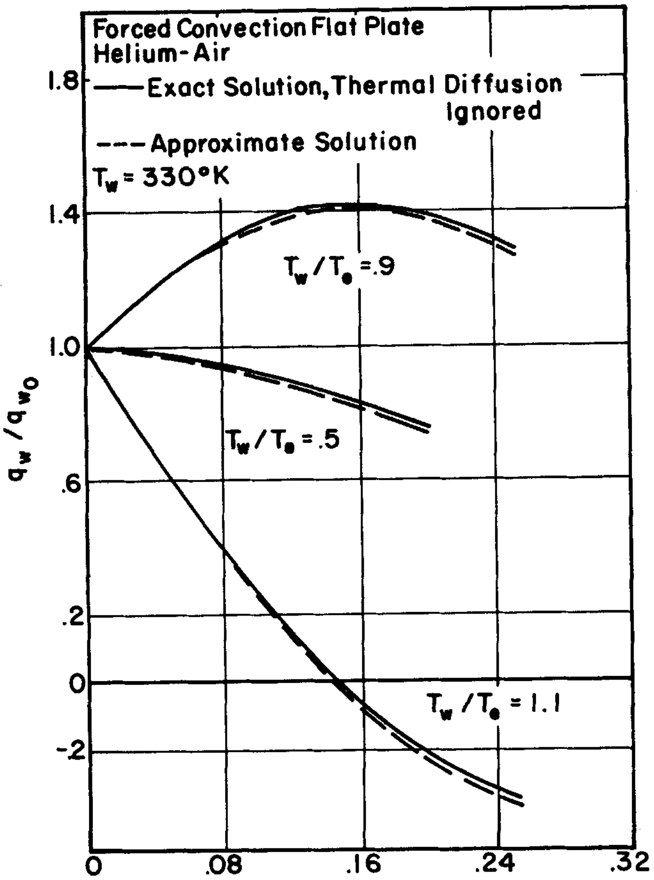

Fig. 3. Comparison of exact and approximate calculation of the ratio of heat transfer with blowing to that without blowing for helium injection into air in flow over a flat plate with diffusion thermo included.

(14), and (33) simultaneously and then uses the results to solve Equation (35). It has been shown (*21*) that negligible error is introduced by this procedure. Furthermore it will be found that solutions for $(T_w/T_e) = 1$, which causes all properties to be independent of temperature, can be used to yield heat transfer results that are quite accurate throughout the entire range of (T_w/T_e) where diffusion thermo is important.

Now the energy equation for θ_2 corresponds to the variable-property problem in which viscous dissipation and thermodynamic coupling are neglected. Within the limits wherein the solutions to Equations (33) and (35) are invariant with (T_w/T_e), one thereby linearizes the effects of diffusion thermo and obtains solutions for arbitrary (T_w/T_e). As indicated previously, the quantity of primary concern here is the heat flux, which now can be written as

$$q_w = -\sqrt{\frac{U_e}{2\nu_e x}}\, T_e k_e \frac{\Lambda_{kw}}{\Lambda_{\mu w}}\, \theta_2'(0) \left[\left(\frac{T_w}{T_e} - 1 \right) - \frac{1}{2}\left(\frac{T_w}{T_e} + 1 \right) \frac{R\mu_e}{k_e}\, \sigma - \left(\frac{T_w}{T_e} - 1 \right) N_{E\sigma VD} \right] \quad (38)$$

where

$$\sigma = -\frac{\dfrac{\Lambda_{kw}}{\Lambda_{\mu w}}\, \theta'_{DT}(0) + \Gamma_2(0)}{\dfrac{\Lambda_{kw}}{\Lambda_{\mu w}}\, \theta_2'(0)}, \quad \sigma_{VD} = -\frac{\theta_2'(0)}{\theta'_{VD}(0)} \quad (39)$$

RESULTS

Effect of Diffusion Thermo on Heat Transfer

The quantities $\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta'_2(0)$ and σ are plotted in Figures 2*a* and 2*b* vs. the interfacial mass flux divided by the molecular weight of the injected gas for various values of (T_w/T_e) for the helium-air system. It is seen that even for ratios varying from 0.5 to 1.1, the results are quite insensitive to (T_w/T_e). When one considers the quantity σ, however, it is seen that the curve for $(T_w/T_e) = 0.5$ starts to deviate from the other curves as the mass flux increases. This is not a serious practical problem however, since the importance of diffusion thermo diminishes as (T_w/T_e) deviates from unity, so the error in evaluating σ becomes less important. Thus, a single solution, say, at $(T_w/T_e) = 1$, can be used in Equation (38) to evaluate the heat flux to give quite accurate results for the range of (T_w/T_e) considered. To demonstrate, the ratio q_w/q_{w_0} is plotted vs. $F(0)$ in Figure 3 for various values of (T_w/T_e). Agreement between the approximate solution (*dotted lines*) and the exact solution (*solid lines*) is seen to be good. The necessary values of the parameter $(R\mu_e/k_e)$ for air as a function of temperature are provided in Figure 4.

Data for the hydrogen-air and carbon dioxide-air systems are also included in Figures 2*a* and 2*b*. It is clear that the effects of diffusion thermo will be negligible in the latter case. One of the interesting aspects of Figures 2*a* and 2*b* is that the helium and hydrogen results correlate very well in terms of the parameter $F(0)/M_A$. This dependence on the pure injected gas molecular weight, M_{H_2} and M_{He}, respectively for the hydrogen-air and helium-air mixtures is markedly different from that of the Gross correlation, which employs one-third power of the injected gas molecular weight. The gross correlation seems to be quite good for the cases of hydrogen and carbon dioxide but does not represent the helium results well.

Use of the Adiabatic Wall Temperature to Correlate Heat Transfer Results

The adiabatic wall temperature T_{aw} is that T_w for which the heat transfer at the wall q_w is zero. From Equation

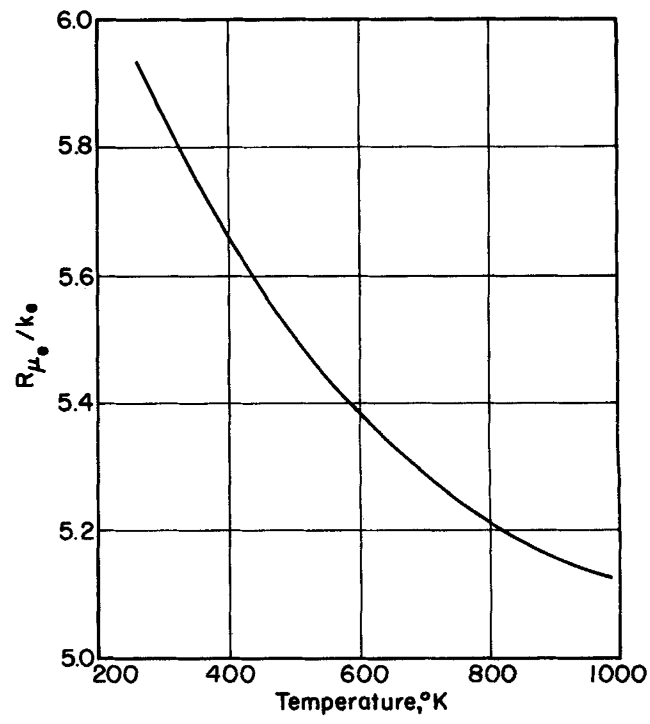

Fig. 4. Variation of the parameter $R\mu_e/k_e$ with temperature for air.

(38) it follows that T_{aw} is given by

$$\frac{T_{aw}}{T_e} = 1 + \frac{\frac{R\mu_e}{k_e}\sigma + \left(\frac{T_w}{T_e} - 1\right) N_E\, \sigma_{VD}}{1 - \frac{1}{2}\frac{R\mu_e}{k_e}\sigma} \tag{40}$$

As noted earlier, previous workers found that the effects of diffusion thermo on the heat transfer coefficient were markedly reduced if the coefficient is defined in terms of the adiabatic wall temperature; that is, $(T_w - T_{aw})$ is used to represent the thermal driving force rather than $(T_w - T_e)$. This suggests that the diffusion thermo term can be treated essentially as an inhomogeneity in the energy equation. However, the diffusion thermo term is not truly a function of independent variables only as evidenced by the presence of the quantity $(T/T_e) = 1 + \left(\frac{T_w}{T_e} - 1\right)\theta$ in the term and by the fact that the variable property case is being considered, unless $(T_w/T_e) = 1$. Division of the diffusion thermo term by an average temperature ratio, accomplished via Equation (32), reduces the effect of the quantity (T/T_e), but there will still be some error involved in attempting to eliminate diffusion thermo effects from the heat transfer coefficient by defining the coefficient in terms of the adiabatic wall temperature. However, the magnitude of this error can be estimated as shown below. For simplicity, viscous dissipation will not be included since we wish to isolate the error related to diffusion thermo.

If a Nusselt number is defined using $(T_w - T_{aw})$ as the driving force, one obtains

$$N_{Nu_x} = \frac{hx}{k_e} = \frac{q_w x}{k_e(T_w - T_{aw})} \tag{41}$$

Substituting for q_w this becomes

$$\sqrt{\frac{2\nu_e}{U_e x}}\, N_{Nu_x} = -\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta_2'(0)\left\{1 + \frac{1}{\frac{T_w}{T_e} - \frac{T_{aw}}{T_e}}\left[\left(\frac{T_{aw}}{T_e} - 1\right) - \frac{1}{2}\left(\frac{T_w}{T_e} + 1\right)\frac{R\mu_e}{k_e}\sigma\right]\right\} \tag{42}$$

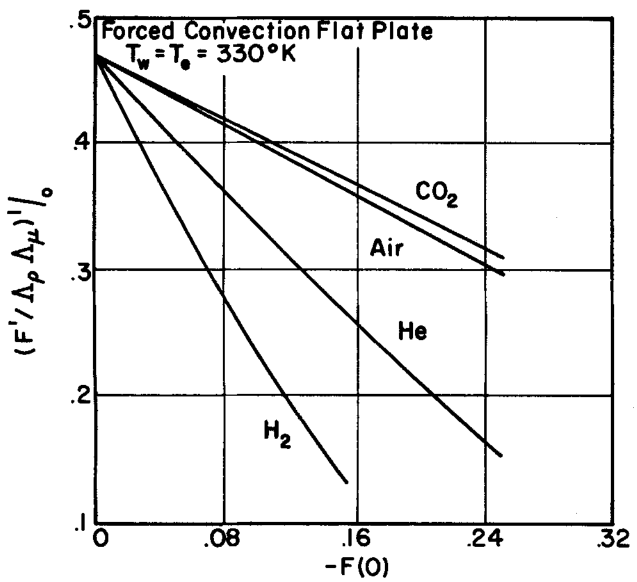

Fig. 5a. Variation of skin friction with interfacial mass flux for various binary systems in flow over a flat plate.

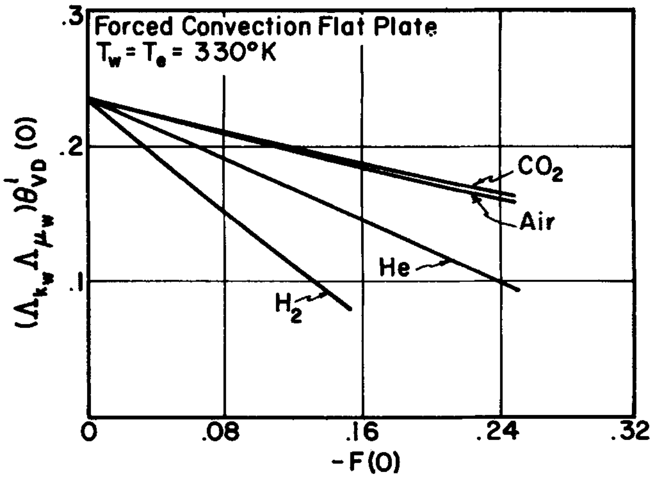

Fig. 5b. Variation of aerodynamic heating term with interfacial mass flux for various binary systems in flow over a flat plate.

With the relation for T_{aw}/T_e given in Equation (40), this becomes

$$\sqrt{\frac{2\nu_e}{U_e x}}\, N_{Nu_x} = -\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta_2'(0)\left\{1 + \frac{1}{\frac{T_w}{T_e} - \frac{T_{aw}}{T_e}}\,\frac{1}{2}\frac{R\mu_e}{k_e}\left[\left(\frac{T_{aw}}{T_e} + 1\right)\sigma_{aw} - \left(\frac{T_w}{T_e} + 1\right)\sigma\right]\right\} \tag{43}$$

where σ_{aw} is the σ obtained when the energy equation is solved for the case of an adiabatic wall. It has been shown previously that σ is primarily a function of the blowing rate, therefore, to a good approximation $\sigma_{aw} = \sigma$ and Equation (43) becomes

$$\sqrt{2}\, N^{-1/2}_{Re_x} N_{Nu_x} = -\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta_2'(0) + \frac{1}{2}\frac{R\mu_e}{k_e}\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta_2'(0)\,\sigma \tag{44}$$

where

$$N_{Re_x} = \frac{U_e x}{\nu_e}$$

The first term is just that which would be obtained if diffusion thermo were ignored completely in solving the energy equation and if the Nusselt number were defined by using $(T_w - T_e)$ as the driving force. The second term represents, to a good approximation, the error involved in assuming that the effects of diffusion thermo are eliminated from the heat transfer coefficient by defining it in terms of the adiabatic wall temperature; that is the error involved in assuming the diffusion thermo term to be an inhomogeneity in the energy equation. It is interesting to note that for a given blowing rate, the percentage error is relatively constant, regardless of the value of (T_w/T_e) and the importance of diffusion thermo. For example, Equation (44) may be rewritten as

$$\sqrt{2}\, N^{-1/2}_{Re_x} N_{Nu_x} = -\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\,\theta_2'(0)\left[1 - \frac{1}{2}\frac{R\mu_e}{k_e}\sigma\right]$$

Since as shown earlier, σ is essentially independent of the ratio (T_w/T_e) and of the temperature level T_e for a given blowing rate, the percentage error is essentially independent of (T_w/T_e) and varies with T_e only via the ratio (μ_e/k_e). Furthermore, since σ increases with blowing over most of the blowing range, so does the percentage error. Since the correlation of Gross et al. (8) can be used to

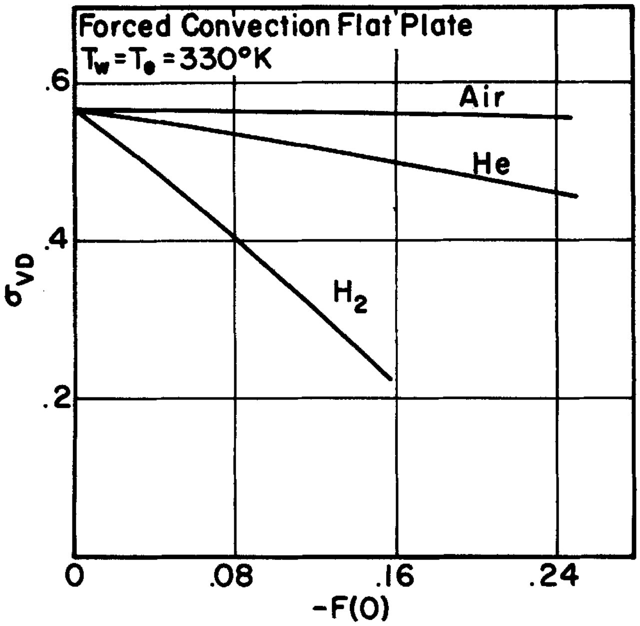

Fig. 6a. σ_{VD} **vs.** $F(0)$ **for helium and hydrogen injection into air in forced convection flow over a flat plate.**

estimate $\theta_2'(0)$, in terms of the value of $\theta_2'(0)$ for the case of $F(0) = 0$, this result provides a convenient correlation for flows over slender bodies. That is, one can obtain a reasonable estimate for the variable-property problem, including thermodynamic coupling effects, in terms of constant property results.

Effect of Viscous Dissipation

The contribution of dissipation to heat transfer at the solid surface is proportional to $(k_w/\mu_w)\ \theta'_{VD}(0)$ and this quantity is plotted vs. the mass flux at the wall for various binary systems in Figure 5*b*. The injection of low molecular weight gases appears to be most effective in reducing dissipation. Since viscous dissipation results from the existence of velocity gradients, the behavior of the skin friction with injection should indicate the corresponding behavior of dissipation, at least for the case of a flat plate with no external pressure gradients. Comparison of Figures 5*a* and 5*b* shows that this is indeed so.

Since viscous dissipation is the major source of boundary-layer heating in high-speed flow over a plate, which is characteristic of slender bodies, when T_w and T_e are close in magnitude it is of value to compare the extent of this heating to the heat flux due to diffusion thermo.

It is easy to show that

$$\left(\frac{T_w}{T_e} - 1\right) N_E = (\gamma_e - 1)\ Ma_e^2$$

where Ma is the Mach number and γ is the ratio of the heat capacity at constant pressure to that at constant volume C_p/C_v.

With Equation (38), the expression for q_w becomes

$$q_w = \sqrt{\frac{U_e}{2\nu_e x}}\ k_e T_e \frac{\Lambda_{kw}}{\Lambda_{\mu w}}\ \theta'_2(0) \left[\left(\frac{T_w}{T_e} - 1\right) - \frac{1}{2}\left(\frac{T_w}{T_e} + 1\right)\frac{R\mu_e}{k_e}\sigma - (\gamma_e - 1)\ Ma^2\ \sigma_{VD}\right] \quad (45)$$

where σ is defined in Equation (39) and

$$\sigma_{VD} = -\frac{\theta'_2(0)}{\theta'_{VD}(0)} \quad (46)$$

The second and third terms in Equation (45) denote the relative contributions of diffusion thermo and viscous dissipation, respectively, to interfacial heat transfer. It is clear that these two terms become more important as $(T_w/T_e) \to 1$.

Of the systems considered in this study only the helium-air and hydrogen-air systems exhibited significant diffusion thermo effects. As may be seen in Figure 2*b*, for these two systems σ increases almost linearly with the interfacial mass flux, being of course zero when $F(0) = 0$. The quantity σ_{VD} is plotted vs. the blowing rate in Figure 6*a* for injection into air of helium, hydrogen, and air. It is seen that the relative importance of viscous dissipation decreases in each case, the decrease being largest for hydrogen, helium, and air, in that order. Actually, for air injection the decrease is so slight as to be negligible.

For given values of T_w, T_e, and $F(0)$, the magnitude of the diffusion thermo term in Equation (45) is fixed. The magnitude of the viscous dissipation term, however, is not fixed until the Mach number is specified. In Figure 6*b* the value of the Mach number at which the diffusion thermo effects and viscous dissipation effects are equal is plotted vs. the blowing parameter for injection into air of hydrogen and helium, with $T_w = T_e = 330°K$. The value of Ma_e in this case is given by

$$Ma_e = \left[\frac{R\mu_e}{k_e(\gamma_e - 1)}\frac{\sigma}{\sigma_{VD}}\right]^{1/2} \simeq \sqrt{16.1\frac{\sigma}{\sigma_{VD}}}$$

It is apparent that the effects of diffusion thermo may be significant even in reasonably high-speed flows. For example, for the case of hydrogen injection with $F(0) = -0.15$ the effects of diffusion thermo are equal to those of dissipation when $Ma_e \simeq 0.35$. It follows that the effects of diffusion thermo will still be one quarter as great as those of dissipation when $Ma_e \simeq 0.7$.

Equation (45) gives the general expression for the heat flux in terms of:

1. Specified free stream properties U_e, ν_e, k_e, T_e, γ_e, Ma_e.
2. The variable properties solutions $\frac{\Lambda_{kw}}{\Lambda_{\mu w}}\ \theta_2'(0)$ obtained by neglecting dissipation and diffusion thermo.
3. The wall temperature ratio (T_w/T_e).

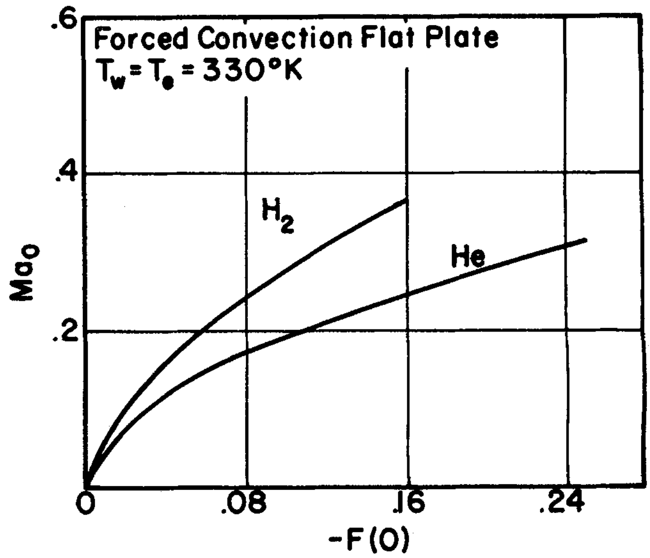

Fig. 6b. Mach number at which the effects of aerodynamic heating and diffusion thermo on heat transfer are equal in magnitude vs. $F(0)$ **for helium and hydrogen injection into air in forced convection flow over a flat plate.**

4. The σ functions.

Once the free stream properties, the mass flux $F(0)$, and the ratio (T_w/T_e) are specified, then exact values of $(\Lambda_{k_w}/\Lambda_{\mu_w})\ \theta_2'(0)$ can be obtained from Figure 2*a* or a reasonable approximation of $(\Lambda_{k_w}/\Lambda_{\mu_w})\ \theta_2'(0)$ is given by the correlation of Gross et al. The σ functions are given in Figures 2*b* and 6*a* as functions of $F(0)$. This constitutes all the information necessary to use Equation (45) to determine the heat flux q_w.

CONCLUSIONS

1. The simple approximate method given to account for the effect of diffusion thermo yields general results which are found on comparison with exact solutions to be accurate enough for most practical purposes.

2. To explain the role of the adiabatic wall temperature in correlating the effects of thermodynamic coupling on heat transfer, the diffusion thermo term is considered as an inhomogeneity or forcing function and thereby is used to generate an approximate particular solution. Perhaps the most important aspect of this approach is that it enables one to obtain a simple accurate error estimate for the adiabatic wall temperature correlation, in terms of the σ function, as given in Equation (44).

3. Viscous dissipation can be reduced significantly by the injection of low molecular weight gases into flows over slender bodies. A quantitative estimate of this reduction is given here in terms of the σ_{VD} function. Also, comparison of the magnitudes of the effects of diffusion thermo and viscous dissipation on heat transfer shows that diffusion thermo may be of significance even in reasonably high-speed flows. In the case of hydrogen injection into air streams with injection rates characterized by $F(0) = -0.15$, the effects of diffusion thermo are on the order of one quarter of those of dissipation for free stream Mach numbers as high as 0.7.

ACKNOWLEDGMENT

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NOTATION

C_f = coefficient of friction, Equation (28)
C_p = heat capacity, for example, cal./(g.) (°C.)
D = coefficient of diffusion
F = stream function, Equation (10)
h = heat transfer coefficient
h_D = mass transfer coefficient
j = mass flux by diffusion, Equation (5*a*)
k = thermal conductivity
K_b = Boltzmann constant
M = molecular weight
Ma = Mach number
N_E = Eckert number, Equation (19)
N_{Nu} = Nusselt number, Equation (26)
N_{Pr} = Prandtl number, $\mu C_p/k$
N_{Sc} = Schmidt number, $\mu/D\rho$
N_{Sh} = Sherwood number, Equation (27)
R = gas constant = 1.987, cal./(g.-mole) (°K.)
T = temperature, absolute
u = x component of velocity
U_e = velocity at edge or boundary layer
v = y component of velocity
x = distance along surface
y = distance normal to surface

Greek Letters

α = thermal diffusion factor, see references 3 and 4
γ = variable as defined by Equation (39)
Γ = variable as defined by Equation (18)
Γ_2 = variable as defined by Equation (37)
Δ = variable as defined by Equation (16)
ϵ = force constant used in evaluation of transport properties
η = variable as defined by Equation (9)
θ = $(T - T_e/(T_w - T_e)$
θ_1 = variable as defined by Equation (23)
θ_2 = variable as defined by Equation (32)
θ_{DT} = variable as defined by Equation (32)
θ_{VD} = variable as defined by Equation (23)
Λ = ratio of property to its value in free stream; for example, $\Lambda_\rho = \rho/\rho_e$, $\Lambda_\mu = \mu/\mu_e$, $\Lambda_k = k/k_e$
Λ_{CAB} = $(C_{pA} - C_{pB})/Cp_e$
μ = viscosity
ν = kinematic viscosity
ρ = density
σ_F = force constant used in evaluation of transport properties
σ = function as defined by Equation (39)
τ_{VD} = function as defined by Equation (39)
ϕ = (ω_A/ω_{A_w})
ψ = stream function as defined by Equation (8)
ω = mass fraction
Ω = variable as defined by Equation (17)

Subscripts

aw = condition of an adiabatic wall
A = component A
B = component B
x = x direction
y = y direction
w = conditions at surface
e = conditions at edge of boundary layer

LITERATURE CITED

1. Baron, J. R., *Am. Rocket Soc. J.*, 1053 (July, 1962).
2. ———, *Intern. J. Heat Mass Transfer*, **6**, 1025 (1963).
3. Chapman, Sidney, and T. G. Cowling, "The Mathematical Theory of Non-Uniform Gases," Cambridge Univ. Press, London (1960).
4. Eckert, E. R. G., and R. M. Drake, "Heat and Mass Transfer," McGraw-Hill, New York (1959).
5. Eckert, E. R. G., W. J. Minkowycz, E. M. Sparrow, and W. E. Ibele, *Intern. J. Heat Mass Transfer*, **6**, 245 (1963).
6. Eckert, E. R. G., P. J. Schneider, A. A. Hayday, and R. M. Larson, *Jet Propulsion*, **28**, 34 (1958).
7. Gollnick, A. F., Jr., *J. Aerospace Sci.*, **29**, 583 (1962).
8. Gross, J. F., J. P. Hartnett, D. J. Masson, and Carl Gazley, Jr., *Intern. J. Heat Mass Transfer*, **3**, 198 (1961).
9. Hirschfelder, J. O., C. F. Curtiss, and R. B. Bird, "Molecular Theory of Gases and Liquids," Wiley, New York (1954).
10. Hurley, D. G., *Aeronaut. Quart.*, **12**, 165 (1961).
11. Sparrow, E. M., "Proceedings of the 1964 Heat Transfer and Fluid Mechanics Institute," W. H. Giedt and S. Levy, ed., Stanford Univ. Press, Calif. (1964).
12. ———, W. J. Minkowycz, and E. R. G. Eckert, *Am. Inst. Aeronaut. Astronaut J.*, **3**, 1341 (1965).
13. ———, *ibid.*, **2**, 652 (1964).
14. ———, *J. Heat Transfer*, **86**, 508 (1964).
15. ———, and W. E. Ibele, *ibid.*, **86**, 311 (Aug., 1964).
16. Sparrow, E. M., C. J. Scott, R. J. Forstrom, and W. A. Ebert, private communication.
17. Tewfik, O. E., *Intern. J. Heat Mass Transfer*, **7**, 409 (1964).
18. ———, E. R. G. Eckert, and L. S. Jurewicz, *Am. Inst. Aeronaut Astronaut J.*, **1**, 1537 (1963).
19. Tewfik, O. E., E. R. G. Eckert, and C. J. Shirtliffe, "Proceedings of the 1962 Heat Transfer and Fluid Mechanics Institute," Stanford Univ. Press, Calif. (1962).
20. Tewfik, O. E., and J. W. Yang, *Intern. J. Heat Mass Transfer*, **6**, 915 (1963).
21. Zeh, D. W., Ph.D. thesis, Syracuse Univ., N. Y. (1966).

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